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GENERATION OF ANALYTIC SEMIGROUPS IN THE L^p TOPOLOGY BY ELLIPTIC OPERATORS IN $\mathbf{R}^n$†

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ABSTRACT

Strongly elliptic differential operators with (possibly) unbounded lower order coefficients are shown to generate analytic semigroups of linear operators on $L^p(\mathbf{R}^n)$, $1 \leqq p \leqq \infty$. An explicit characterization of the domain is given for $1 < p < \infty$. An application to parabolic problems is also included.

1. Introduction and notation

This paper is concerned with the analysis of some spectral properties of second order elliptic operators

$$Eu = \sum_{ij} a_{ij} D_{ij} u + \sum_j b_j D_j u - cu,$$

where a_{ij}, b_j, c are real-valued functions defined on $\mathbf{R}^n$. The (possible) growth of the coefficients of E will be referred to a differentiable function $V(x) \geqq 1$ such that

$$|DV(x)| = o([V(x)]^{3/2}) \tag{1.1}$$

(here and in the sequel, by $f = o(q)$ we mean that for any $\varepsilon > 0$ there exists k_ε such that $|f(x)| \leqq \varepsilon g(x) + k_\varepsilon$). In fact, we will impose conditions of the form

$$a_{ij},\ b_j/\sqrt{V},\quad c/V \in C(\mathbf{R}^n)_u, \qquad \forall\, i, j = 1, \ldots, n,$$

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where $C(\mathbf{R}^n)_u$ (resp. $C(\mathbf{R}^n)$) denotes the space of uniformly continuous (resp. continuous) bounded functions defined on $\mathbf{R}^n$.

The spectral properties we are concerned with are estimates that imply the generation of analytic semigroups in $L^p(\mathbf{R}^n)$, $1 \leqq p \leqq \infty$. Moreover, these estimates give a characterization of the domain of E in the case of $1 < p < \infty$.

Similar problems have been studied by many authors. As for the case of $1 < p < \infty$, Agmon [1] studied operators defined on bounded domains (for this case see also [3]). Higouchi [12] and Freeman and Schechter [1] proved L^p estimates for operators with uniformly bounded coefficients. For operators with bounded coefficients, generation in L^1 was treated by Amann [2] and Pazy [16]; Davies and Simon [9], Davies [8] and Voigt [17] analyzed Schroedinger operators with unbounded potential term.

Generation results in different topologies, such as the Holder topology, are proved in [5] under stronger assumptions on V.

We note that, if one is just interested in the semigroup generation, then one may assume less restrictive conditions than (1.1) (see for instance [15] where c is supposed to be bounded below; see also Section 2 of [5]).

The plan of the paper is the following: in Section 2 we obtain a preliminary result in L^p, that we apply to get the generation theorem in the uniform topology (see Section 3). Then, by duality techniques, the L^1 estimates follow in Section 4. Thus, by interpolating between L^1 and L^∞, we obtain the L^p estimates under general assumptions (see Section 5). Finally, in Section 6 of this paper we use a maximal regularity result for the abstract Cauchy problem to study a parabolic equation with right-hand side in $L^p(0, T; L^q(\mathbf{R}^n))$.

We conclude this section introducing the main notation of the paper. If $x_0 \in \mathbf{R}^n$ and $r > 0$ we set

$$B(x_0, r) = \{x \in \mathbf{R}^n : |x - x_0| < r\}.$$

Let $V(x) \geqq 1$ be a measurable function and $1 \leqq p < \infty$. For $j = 0, 1, 2$ we denote by $H^{j,p}(\mathbf{R}^n, V)$ the space of functions u such that

$$\| u \|_{j,p,V}^p = \sum_{|\beta| \leqq j} \int_{\mathbf{R}^n} V^{(j-|\beta|)p/2} |D^\beta u|^p dx < +\infty.$$

For simplicity, we drop the subscript or superscript p when $p = 2$.

With a slight abuse of notation, we denote by $C^1(\mathbf{R}^n, V)$ the space of functions u such that

$$D_i u, \quad Vu \in C(\mathbf{R}^n), \qquad \forall i = 1, \dots, n$$

and set

$$\| u \|_{1,\infty,V} = |u|_{1,\infty} + \| Vu \|_{0,\infty}, \quad |u|_{1,\infty} = \left\| \left[\sum_j |D_j u|^2 \right]^{1/2} \right\|_{0,\infty},$$

$$\| u \|_{0,\infty} = \sup_{x \in \mathbf{R}^n} |u(x)|.$$

For the sake of brevity, in all the notation above we omit specifying the weight V if $V \equiv 1$. Furthermore, we set

$$C^1(\mathbf{R}^n)_u = \{ u \in C(\mathbf{R}^n)_u : D_i u \in C(\mathbf{R}^n)_u, \ \forall i = 1, \ldots, n \}.$$

2. A preliminary result in $L^p(\mathbf{R}^n)$, $2 < p < \infty$

We consider an elliptic operator

$$Eu = \sum_{ij} a_{ij} D_{ij} u + \sum_j b_j D_j u - cu \tag{2.1}$$

where a_{ij}, b_j, c are real-valued functions defined on $\mathbf{R}^n$. In this section we will refer the growth of the coefficients of E to a differentiable function $V(x) \geqq 1$ such that

$$|DV(x)| \leqq kV(x), \qquad \forall x \in \mathbf{R}^n \tag{2.2}$$

for some constant $k > 0$. In fact, we assume that

$$a_{ij}, b_j/\sqrt{V}, \quad c/V \in C(\mathbf{R}^n)_u, \qquad \forall i, j = 1, \ldots, n. \tag{2.3}$$

In particular, we suppose that

$$c(x) = \alpha(x)V(x) + \beta(x), \qquad \forall x \in \mathbf{R}^n \tag{2.4}$$

where $\alpha, \beta \in C(\mathbf{R}^n)_u$, $\alpha(x) \geqq \alpha_0 > 0$, and that E is uniformly elliptic, i.e.

$$\sum_{ij} a_{ij}(x) \xi_i \xi_j \geqq \nu |\xi|^2, \qquad \forall x, \xi \in \mathbf{R}^n \tag{2.5}$$

for some constant $\nu > 0$. Finally, we assume that

$$\sum_j |b_j(x)|^2 \leqq \nu \alpha_0 B^2 V(x), \qquad \forall x \in \mathbf{R}^n \tag{2.6}$$

for some constant $B \in [0, 2[$.

Under the assumptions above, it is known that operator E is the infinitesimal generator of an analytic semigroup on both $L^2(\mathbf{R}^n)$ and $L^\infty(\mathbf{R}^n)$, see

Theorems 1.2 and 1.4 in [5]. In particular, there exists $\omega_1 \in \mathbf{R}$ such that, if $\lambda \in \mathbf{C}$ and $\operatorname{Re}\lambda \geqq \omega_1$, then for each $f \in L^2(\mathbf{R}^n)$ the equation

$$(\lambda - E)u = f \tag{2.7}$$

has a unique solution $u \in H^2(\mathbf{R}^n, V)$ and

$$\| u \|_{2,V} + |\lambda - \omega_1|^{1/2} \| u \|_{1,\sqrt{V}} + |\lambda - \omega_1| \, \| u \|_0 \leqq k \| f \|_0 \tag{2.8}$$

Moreover, if $f \in L^\infty(\mathbf{R}^n)$, then there exists $\omega_2 \geqq \omega_1$ such that for any $\lambda \in \mathbf{C}$ satisfying $\operatorname{Re}\lambda \geqq \omega_2$ equation (2.7) has a unique solution $u \in C^1(\mathbf{R}^n, \sqrt{V}) \cap H^2(\mathbf{R}^n)_{\mathrm{loc}}$ and

$$|\lambda - \omega_2|^{1/2} \| u \|_{1,\infty,\sqrt{V}} + |\lambda - \omega_2| \, \| u \|_{0,\infty} \leqq k \| f \|_{0,\infty}. \tag{2.9}$$

We now want to show a similar result in the L^p case.

2.1. THEOREM. *Assume* (2.2)–(2.6) *and let* $a_{ij} \in C^1(\mathbf{R}^n)_u$. *There exists* $\omega_3 \in \mathbf{R}$ *such that, if* $\operatorname{Re}\lambda \geqq \omega_3$, *then for each* $f \in L^p(\mathbf{R}^n)$, $2 \leqq p < +\infty$, *equation* (2.7) *has a unique solution* $u \in H^{1,p}(\mathbf{R}^n, \sqrt{V}) \cap H^2(\mathbf{R}^n)_{\mathrm{loc}}$ *and*

$$\| Vu \|_{0,p} + |\lambda - \omega_3|^{1/2} \| u \|_{1,p,\sqrt{V}} + |\lambda - \omega_3| \, \| u \|_{0,p} \leqq k \| f \|_{0,p} \tag{2.10}$$

where k is independent of the derivatives of the coefficients a_{ij}, *as well as of p and* λ.

PROOF. Consider the operator in divergence form

$$E_* u = \sum_{ij} D_i(a_{ij} D_j u) - \sum_{ij} (D_i a_{ij})(D_j u) + \sum b_j D_j u - cu$$

and set

$$A = \max_{ij} \| a_{ij} \|_{1,\infty}. \tag{2.11}$$

By interpolating between (2.8) and (2.9) it follows that, if $\operatorname{Re}\lambda \geqq \omega_2$, then for each $p \in]2, +\infty[$ the equation

$$(\lambda - E_*)u = f \in L^p(\mathbf{R}^n)$$

has a unique solution $u \in H^{1,p}(\mathbf{R}^n, \sqrt{V}) \cap H^2(\mathbf{R}^n)_{\mathrm{loc}}$ and

$$|\lambda - \omega_3|^{1/2} \| u \|_{1,p,\sqrt{V}} + |\lambda - \omega_3| \, \| u \|_{0,p} \leqq k \| f \|_{0,p}. \tag{2.12}$$

Let now $\{x_N\}_{N \geqq 1}$ be the sequence of points with integer coordinates in $\mathbf{R}^n$ so that $\bigcup_N B(x_N, \sqrt{n})$ covers $\mathbf{R}^n$ and $\theta_N \in C^\infty(\mathbf{R}^n)$ be a cut-off function such that

$0 \leqq \theta_N \leqq 1$, $\theta_N \equiv 1$ on $B(x_N, \sqrt{n})$, $\theta_N \equiv 0$ out of $B(x_N, 2\sqrt{n})$ and $|D^\alpha \theta_N| \leqq k$ for $|\alpha| \leqq 2$ and $N \geqq 1$. Obviously, $u_N = \theta_N u$ is a solution of the equations

$$(\lambda - E)u_N = g_N \quad \text{and} \quad (\lambda - E_*)u_N = g_N$$

where, in view of (2.12),

$$g_N = \theta_N f - \sum_{ij} a_{ij}(2D_i u D_j \theta_N + u D_{ij}\theta_N) - \sum_j b_j u D_j \theta_N \in L^p(\mathbf{R}^n). \tag{2.13}$$

Therefore, we can derive L^p estimates by Pazy's technique. Define

$$u_N^* = |u_N|^{p-2}\bar{u}_N \in H^{1,q}(\mathbf{R}^n, \sqrt{V}) \qquad \text{where } q = (p-1)/p$$

(in the sequel we omit the subscript N to abbreviate the notation). Then

$$\int_{\mathbf{R}^n} (\lambda - E_*)u\, u^* dx = \int_{\mathbf{R}^n} \sum a_{ij} D_j u D_i u^* dx + \int_{\mathbf{R}^n} \sum D_i a_{ij} D_j u\, u^*$$
$$- \int_{\mathbf{R}^n} \sum b_j D_j u\, u^* dx + \int_{\mathbf{R}^n} (c + \lambda) u\, u^* dx.$$

Now, arguing as in [16] (p. 215),

$$\operatorname{Re} \int_{\mathbf{R}^n} \sum a_{ij} D_j u D_i u^* dx \geqq \nu \int_{\mathbf{R}^n} \sum |D_j u|^2 |u|^{p-2} dx.$$

Moreover, recalling (2.6) and (2.12),

$$\int_{\mathbf{R}^n} \sum |b_j D_j u\, u^*| dx \leqq \nu B/2 \int_{\mathbf{R}^n} \sum |D_j u|^2 |u|^{p-2} dx + B/2 \int_{\mathbf{R}^n} V|u|^p dx,$$

$$\int_{\mathbf{R}^n} \sum |D_i a_{ij} D_j u\, u^*| dx \leqq k_A \int_{\mathbf{R}^n} |u|^p dx + (1/2 - B/4)\nu \int_{\mathbf{R}^n} \sum |D_j u|^2 |u|^{p-2} dx.$$

Therefore, choosing

$$\omega_3 = \omega_2 + 2k_A + 1 \tag{2.14}$$

we obtain

$$\int_{\mathbf{R}^n} |gu^*| dx \geqq \operatorname{Re} \int_{\mathbf{R}^n} (\lambda - E_*) u\, u^* dx$$
$$\geqq (1/2 - B/4)\nu \int_{\mathbf{R}^n} \sum |D_j u|^2 |u|^{p-2} dx$$
$$+ (1 - B/2) \int_{\mathbf{R}^n} V|u|^p dx + \int_{\mathbf{R}^n} |u|^p dx$$

and so

$$\int_{\mathbf{R}^n} |g_N|\,|u_N|^{p-1}dx \geqq (1-B/2)\int_{\mathbf{R}^n} V|u_N|^p dx. \tag{2.15}$$

Now, notice that (2.2) implies that

$$V(x) \leqq \mu V(y) \quad \text{for } |x-y| \leqq 4\sqrt{n} \tag{2.16}$$

for some constant $\mu > 0$. Then, (2.15) and (2.16) yield, for any $N \geqq 1$,

$$\| g_N \|_{0,p}(\| u_N \|_{0,p})^{p-1} \geqq (1-B/2)\mu^{-1}V(x_N)(\| u_N \|_{0,p})^p.$$

Hence

$$\| g_N \|_{0,p} \geqq (1-B/2)\mu^{-1}V(x_N)\| u_N \|_{0,p} \geqq (1-B/2)\mu^{-2}\| Vu_N \|_{0,p}$$

and finally

$$\int_{\mathbf{R}^n} |Vu|^p dx \leqq \int_{\mathbf{R}^n} \sum_N |Vu_N|^p dx \leqq \mu^{2p}(1-B/2)^{-p}\int_{\mathbf{R}^n} \sum_N |g_N|^p dx.$$

Thus, (2.10) follows from the last inequality and from (2.12). □

3. Generation in $L^\infty(\mathbf{R}^n)$

Using estimate (2.10), we can improve the bound in the uniform norm obtained in Section 5 of [7]. Let ω_3 be the number given by Theorem 2.1.

3.1. Lemma. *Assume* (2.2)–(2.6) *and let* $a_{ij} \in C^1(\mathbf{R}^n)_u$. *If* $\operatorname{Re}\lambda \geqq \omega_3$, *then, for each* $f \in L^\infty(\mathbf{R}^n)$, *equation* (2.7) *has a unique solution* $u \in C^1(\mathbf{R}^n, \sqrt{V}) \cap H^2(\mathbf{R}^n)_{\mathrm{loc}}$ *and*

$$|\lambda-\omega_3|^{1/2}\| u \|_{1,\infty,\sqrt{V}} + |\lambda-\omega_3|\,\| u \|_{0,\infty} \leqq k_1 \| f \|_{0,\infty}, \tag{3.1}$$

$$\| Vu \|_{0,\infty} \leqq k_1 \| f \|_{0,\infty}, \tag{3.2}$$

where k_1 *is independent of* λ, *as well as of the derivatives of the coefficients* a_{ij}.

Proof. We merely need to prove the bound (3.2), which is not contained in [5]. From (2.10) it follows that (3.2) holds if $f \in L^p(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ for some $p \in\,]2, +\infty[$. In the general case, let $\eta_N \in C^\infty(\mathbf{R}^n)$ be a cut-off function such that $0 \leqq \eta \leqq 1$, $\eta \equiv 1$ on $B(0, N)$, $\eta \equiv 0$ out of $B(0, 2N)$ and define $u_N = \eta_N u$. Then, $(\lambda - E)u_N = g_N \in L^\infty(R^n)$, where

$$g_N = \eta_N f - \sum_{ij} a_{ij}(2D_i u D_j \eta_N + u D_{ij}\eta_N) - \sum_j b_j u D_j \eta_N$$

has compact support. Then, by (3.1),

$$\| Vu_N \|_{0,\infty} \leqq k \| g_N \|_{0,\infty} \leqq k' \| f \|_{0,\infty}$$

which in turn implies (3.2). Notice that, as in Theorem 2.1, ω_3 is the only constant depending on the derivatives of the coefficients (see (2.14)). □

3.2. LEMMA. *Under the assumptions of Lemma* 3.1

$$\left\| \left[V \sum_j |D_j u|^2 \right]^{1/2} \right\|_{0,\infty} \leqq k_2 \| f \|_{0,\infty} \tag{3.3}$$

where k_2 is independent of λ, as well as of the derivatives of the coefficients a_{ij}.

PROOF. Suppose first that x is such that

$$\left[V(x) \sum_j |D_j u(x)|^2 \right]^{1/2} = \left\| \left[V \sum_j |D_j u|^2 \right]^{1/2} \right\|_{0,\infty}.$$

Then, arguing as in the proof of Lemma 5.1 in [5], it follows that for any $\sigma > 0$ there exists $k_\sigma > 0$ such that

$$\begin{aligned} &r^{-1} | u |_{1,\infty,B(x,r/2)} \\ &\quad \leqq k_0(\sigma + k_\sigma r^{-1} V(x)^{-1/2})(\| f \|_{0,\infty} + r^{-2} \| u \|_{0,\infty,B(x,r)} + r^{-1} | u |_{1,\infty,B(x,r),\sqrt{V}}) \end{aligned}$$

for any $r > 0$. Here k_0 is a constant, independent of σ and λ, and

$$\| u \|_{0,\infty,B(x,r)} = \sup_{x \in B(x,r)} | u(x) |$$

(we have defined $| u |_{1,\infty,B(x,r/2)}$ and $| u |_{1,\infty,B(x,r/2),\sqrt{V}}$ accordingly). Now, choose

$$\sigma = 1/(4k_0\mu^2), \quad r = k_\sigma \sigma^{-1} V(x)^{-1/2}$$

where μ is the constant that appears in (2.16). By (2.16) and the last inequality we obtain

$$\begin{aligned} &(4k_0 k_\sigma \mu^3)^{-1} \left\| \left[V \sum_j |D_j u|^2 \right]^{1/2} \right\|_{0,\infty} \\ &\quad \leqq (2\mu^2)^{-1} \left[\| f \|_{0,\infty} + \| Vu \|_{0,\infty} + (4k_0 k_\sigma \mu)^{-1} \left\| \left[V \sum_j |D_j u|^2 \right]^{1/2} \right\|_{0,\infty} \right]. \end{aligned}$$

The last inequality and (3.2) imply (3.3).

Next, assume that $\sqrt{V} u \in C^1(\mathbf{R}^n)$. Then, one may repeat the previous argument about a point x such that $[V(x)\sum_j |D_j u(x)|^2]^{1/2}$ is arbitrarily close to $\| [V\sum_j |D_j u|^2]^{1/2} \|_{0,\infty}$ and (3.3) follows again.

The general case may be treated using the analysis above and the approximation procedure contained in the proof of Theorem 1.4 in [5]. □

Let us now consider the following perturbations of operator E:

$$\tilde{E}u = Eu + \sum_j \tilde{b}_j D_j u + \tilde{c}u. \tag{3.4}$$

The following result is a trivial consequence of Lemmas 3.1 and 3.2.

3.3. LEMMA. *Assume* (2.2)–(2.6) *and let* $a_{ij} \in C^1(\mathbf{R}^n)_u$, $\tilde{b}_j, \tilde{c} \in L^\infty(\mathbf{R}^n)$. *There exists* $\delta > 0$ *such that, if*

$$|\tilde{b}_j| \leqq \delta\sqrt{V} + \tilde{K}, \quad |\tilde{c}| \leqq \delta V + \tilde{K} \qquad \text{with } \tilde{K} \geqq 1 \tag{3.5}$$

and if $u \in C^1(\mathbf{R}^n, \sqrt{V}) \cap H^2(\mathbf{R}^n)_{\mathrm{loc}}$ *is a solution of the equation*

$$(\lambda - \tilde{E})u = f \in L^\infty(\mathbf{R}^n)$$

with $\operatorname{Re}\lambda \geqq \omega_4 = \omega_3 + 2\tilde{K}$, *then*

$$\begin{aligned} &\left\| \left[V\sum_j |D_j u|^2 \right]^{1/2} \right\|_{0,\infty} + \| Vu \|_{0,\infty} + |\lambda - \omega_4|^{1/2} \| u \|_{1,\infty,\sqrt{V}} \\ &\quad + |\lambda - \omega_4| \, \| u \|_{0,\infty} \leqq k_2 \| f \|_{0,\infty}. \end{aligned} \tag{3.6}$$

We now state an estimate which is useful when dealing with functions V satisfying more general growth conditions than (2.2).

3.4. LEMMA. *Let* $V(x) \geqq 1$ *be a differentiable function on* $\mathbf{R}^n$ *satisfying*

$$|DV(x)| \leqq \gamma[V(x) + \Gamma]^{3/2}, \qquad \forall x \in \mathbf{R}^n \tag{3.7}$$

for some constants $\gamma, \Gamma > 0$. *Then*, $\forall h \in \mathbf{R}^n$ *such that* $|h| < 2/\gamma\sqrt{V(x)+\Gamma}$,

$$\begin{aligned} &[V(x)+\Gamma]\left[\left(\frac{2}{\gamma|h|\sqrt{V(x)+\Gamma}+2}\right)^2 - 1\right] \\ &\leqq V(x+h) - V(x) \leqq [V(x)+\Gamma]\left[\left(\frac{2}{\gamma|h|\sqrt{V(x)+\Gamma}-2}\right)^2 - 1\right]. \end{aligned} \tag{3.8}$$

We omit the standard proof of the previous Lemma.

Next, we turn to the analogue of the bounds (3.1), (3.2), (3.3).

3.5. LEMMA. *Assume* (2.3)–(2.6) *and let* $a_{ij} \in C^1(\mathbf{R}^n)_u$. *Let* $V(x) \geqq 1$ *be a differentiable function on* $\mathbf{R}^n$ *satisfying*

$$|DV(x)| = o([V(x)]^{3/2}). \tag{3.9}$$

There exists $\omega_5 \in \mathbf{R}$ *such that, if* $u \in C^1(\mathbf{R}^n, \sqrt{V}) \cap H^2(\mathbf{R}^n)_{\mathrm{loc}}$ *is a solution of the equation*

$$(\lambda - E)u = f \in L^\infty(\mathbf{R}^n)$$

with $\operatorname{Re}\lambda \geqq \omega_5$, *then*

$$\begin{aligned} &\left\| \left[V \sum_j |D_j u|^2 \right]^{1/2} \right\|_{0,\infty} + \| Vu \|_{0,\infty} + |\lambda - \omega_5|^{1/2} \| u \|_{1,\infty,\sqrt{V}} \\ &\quad + |\lambda - \omega_5| \, \| u \|_{0,\infty} \leqq k_3 \| f \|_{0,\infty} \end{aligned} \tag{3.10}$$

where k_3 *is independent of* λ, *as well as of the derivatives of the coefficients* a_{ij}.

PROOF. Let $x_0 \in \mathbf{R}^n$ and set $V_0 = V(x_0)$. Let $\eta > 0$ be a number that will be fixed in the sequel. Define

$$K_0 = \max\{1, \|\alpha\|_{0,\infty} + \|\beta\|_{0,\infty} + \sqrt{\nu\alpha_0}B\}$$

and

$$E_0 u = \sum_{ij} a_{ij}(x) D_{ij} u + \sum_j b_j(x_0) D_j u - c(x_0) u$$

where α, β, ν, α_0, B are the same as in (2.4), (2.5), (2.6). Let $\delta_0 > 0$ be the number given by Lemma 3.3 for operator E_0. We set

$$\gamma = \eta^{-1}[1 - 8\sqrt{K_0/(\delta^2 + 64K_0)}]. \tag{3.11}$$

From condition (3.9) it follows that there exists $\Gamma > 0$ such that

$$|DV(x)| \leqq \gamma[V(x) + \Gamma]^{3/2}, \qquad \forall x \in \mathbf{R}^n. \tag{3.12}$$

Let $r = \eta/\sqrt{V_0 + \Gamma}$ and θ be a cut-off function such that

$$0 \leqq \theta \leqq 1, \quad \theta \equiv 1 \text{ on } B(x_0, r), \quad \theta \equiv 0 \text{ out of } B(x_0, 2r),$$

$$|D^\alpha \theta| \leqq (2\eta/\sqrt{V_0 + \Gamma})^{|\alpha|}, \qquad |\alpha| \leqq 2.$$

We denote by $\mathbf{1}_B(x)$ the characteristic function of a set B. Define

$$\tilde{b}_j(x) = [b_j(x) - b_j(x_0)]\,\mathbf{1}_{B(x_0,2r)}(x) \quad \text{and} \quad \tilde{c}(x) = [c(x) - c(x_0)]\,\mathbf{1}_{B(x_0,2r)}(x).$$

If we set, as in (3.4),

$$\tilde{E}_0 u = E_0 u + \sum_j \tilde{b}_j D_j u - \tilde{c} u,$$

then

$$(3.13)\quad (\lambda - \tilde{E}_0)(\theta u) = \theta f - \sum_{ij} a_{ij}(2D_i u D_j \theta + u D_{ij}\theta) - \sum_j b_j u D_j \theta = F.$$

Now, notice that Γ in (3.12) may be chosen so that $\tilde{E}_0$ satisfies condition (3.5) of Lemma 3.3. Indeed, from (2.3), (2.4) and (2.6) it follows that

$$|c(x)/V(x)| + \left[\sum_j |b_j(x)|^2/V(x)\right]^{1/2} \leqq K_0$$

and

$$\left|\frac{c(x)}{V(x)} - \frac{c(y)}{V(y)}\right| + \sum \left|\frac{b_j(x)}{\sqrt{V(x)}} - \frac{b_j(y)}{\sqrt{V(y)}}\right| \leqq \omega(|x-y|)$$

where $\omega(t) \geqq 0$ is non-decreasing and $\lim_{t\downarrow 0} \omega(t) = 0$. Therefore,

$$|\tilde{c}(x)| \leqq \omega(2\eta/\sqrt{V_0+\Gamma})V_0 + K_0(V_0+\Gamma)[(\gamma\eta-1)^{-2}-1],$$

$$|\tilde{b}_j(x)| \leqq \omega(2\eta/\sqrt{V_0+\Gamma})\sqrt{V_0} + K_0(\sqrt{V_0}+\sqrt{\Gamma})[(\gamma\eta-1)^{-2}-1]^{1/2},$$

and so, choosing Γ_η such that

$$\omega(2\eta/\sqrt{\Gamma_\eta}) < \delta_0/4,$$

recalling (3.11) we have, for $\Gamma \geqq \Gamma_\eta$,

$$|\tilde{c}(x)| \leqq \delta_0 V_0/2 + \Gamma\delta_0/2 \quad \text{and} \quad |\tilde{b}_j(x)| \leqq \delta_0\sqrt{V_0}/2 + \sqrt{\Gamma}\delta_0/2.$$

Next, from Lemma 3.3 we conclude that, if $\operatorname{Re}\lambda \geqq \omega_4$, then

$$\left\| \left[V_0 \sum_j |D_j(\theta u)|^2\right]^{1/2} \right\|_{0,\infty} + \| V_0 \theta u \|_{0,\infty} + |\lambda - \omega_4|^{1/2} \| \theta u \|_{1,\infty,\sqrt{V_0}}$$
$$+ |\lambda - \omega_4| \, \| \theta u \|_{0,\infty} \leqq k_2 \| F \|_{0,\infty},$$

where F is given by (3.13). On the other hand,

$$\| F \|_{0,\infty} \leqq \| f \|_{0,\infty} + 4A_0(V_0+\Gamma)\eta^{-2} \| u \|_{0,\infty} + 8A_0(\sqrt{V_0}+\sqrt{\Gamma})\eta^{-1} | u |_{1,\infty}$$
$$+ 4K_0(\sqrt{V_0}+\sqrt{\Gamma})\eta^{-1} \| \sqrt{V} u \|_{0,\infty}$$

where

$$A_0 = \max_{ij} \| a_{ij} \|_{0,\infty}.$$

Now, choosing

$$\eta = 80(A_0K_2 + 2A_0 + K_0)[1 - 4\sqrt{K_0/(\delta^2 + 64K_0)}]$$

and recalling (3.8) we obtain

$$\begin{aligned} &\left\| \left[V_0 \sum_j |D_j(\theta u)|^2 \right]^{1/2} \right\|_{0,\infty} + \| V_0\theta u \|_{0,\infty} + |\lambda - \omega_4|^{1/2} \| \theta u \|_{1,\infty,\sqrt{V_0}} \\ (3.14) \qquad &+ |\lambda - \omega_4| \, \| \theta u \|_{0,\infty} \\ &\leqq k_4 \| f \|_{0,\infty} + 10^{-1} \left(\| Vu \|_{0,\infty} + \left\| \left[V \sum_j |D_j u|^2 \right]^{1/2} \right\|_{0,\infty} \right) + k_5 \| u \|_{1,\infty,\sqrt{V}}. \end{aligned}$$

Now, suppose that the functions that appear in the left-hand side of (3.10) attain their maximum at x_i, $i = 1, 2, 3, 4$. Then, repeating the previous argument with $x_0 = x_i$, we conclude that (3.10) holds with

$$\omega_5 = \omega_4 + 64(k_5)^2.$$

Finally, in case some of the above maximum points do not exist, one can still obtain (3.10). In fact, one may apply the analysis above about points at which the functions that appear in the left-hand side of (3.10) are arbitrarily close to their suprema. □

We are ready to prove the main result of this section.

3.6. THEOREM. *Assume* (2.3)–(2.6) *and let* $a_{ij} \in C^1(\mathbf{R}^n)_u$. *Let* $V(x) \geqq 1$ *be a differentiable function on* $\mathbf{R}^n$ *satisfying*

$$|DV(x)| = o([V(x)]^{3/2}).$$

If $\operatorname{Re} \lambda \geqq \omega_5$, *then the equation*

$$(\lambda - E)u = f \in L^\infty(\mathbf{R}^n)$$

has a unique solution $u \in C^1(\mathbf{R}^n, \sqrt{V}) \cap H^2(\mathbf{R}^n)_{\mathrm{loc}}$. *Moreover*, (3.10) *holds*.

PROOF. Let $\eta_N \in C^\infty(\mathbf{R}^n)$ be a cut-off function such that $0 \leqq \eta_N \leqq 1$, $\eta_N \equiv 1$ on $B(0, N)$, $\eta_N \equiv 0$ out of $B(0, 2N)$ and define

$$x_N = \eta_N(x)x + N(1 - \eta_N(x))x|x|^{-1},$$

$$E_N = \sum_{ij} a_{ij}(x)D_{ij} + \sum_j b_j(x_N)D_j - c(x_N),$$

$$V_N = V(x_N).$$

Let $u_N \in C^1(\mathbf{R}^n) \cap H^2(\mathbf{R}^n)_{\mathrm{loc}}$ be a solution of the equation

$$(\lambda - E_N)u_N = f \in L^\infty(\mathbf{R}^n), \qquad \operatorname{Re}\lambda \geqq \omega_5.$$

Then, by Lemma 3.5,

$$\left\| \left[V_N \sum_j |D_j u_N|^2 \right]^{1/2} \right\|_{0,\infty} + \| V_N u_N \|_{0,\infty}$$

$$+ |\lambda - \omega_5|^{1/2}(\| u_N \|_{1,\infty} + \| \sqrt{V_N} u_N \|_{0,\infty}) + |\lambda - \omega_5| \, \| u_N \|_{0,\infty} \leqq k_3 \| f \|_{0,\infty}.$$

Now, using some known compactness arguments (see e.g. the proof of Theorem 1.4 in [5]), one can easily obtain the thesis. □

3.7. Remark. As already noted in Remark 2.2 of [7], the above result extends to more general weighted spaces. This generalization is very useful to deal with stochastic partial differential equations (see [6], [7]). Let π be a twice differentiable function on $\mathbf{R}^n$ satisfying

$$\sum_j |D_j \pi(x)|^2 + \sum_{ij} |D_{ij}\pi(x)| = o(V(x)) \tag{3.15}$$

and define

$$\Pi(x) = \exp[\pi(x)].$$

If $\Pi f \in L^\infty(\mathbf{R}^n)$, then (3.10) holds replacing u by Πu.

4. Generation in $L^1(\mathbf{R}^n)$

In this section we consider operators in divergence form

$$Au = \sum_{ij} D_i(a_{ij}D_j u) + \sum_j b_j D_j u - cu.$$

We assume that $V(x) \geqq 1$ is a differentiable function on $\mathbf{R}^n$ satisfying

$$|DV(x)| = o([V(x)]^{3/2}) \tag{4.1}$$

and a_{ij}, b_j and c satisfy (2.3)–(2.6). We also suppose that a_{ij}, b_j and c are differentiable on $\mathbf{R}^n$ and

$$D_k a_{ij},\ D_k b_j / V,\quad D_k c / V^{3/2} \in C(\mathbf{R}^n)_u, \tag{4.2}$$

$$D_k b_j = o(V),\quad D_k c = o(V^{3/2}), \tag{4.3}$$

for all $i, j, k = 1, \ldots, n$.

A first generation result is the following.

4.1. Lemma. *Assume* (4.1), (4.2), (4.3) *and* (2.3)–(2.6). *There exists a constant k_6, independent of the derivatives of the coefficients, and $\omega_6 \in \mathbf{R}$ such that for* $\operatorname{Re}\lambda \geqq \omega_6$ *the equation*

$$(\lambda - A)u = f \in L^1(\mathbf{R}^n)$$

has a unique solution $u \in L^1(\mathbf{R}^n)$ and

$$|\lambda - \omega_6|\ \| u \|_{0,1} \leqq k_6 \| f \|_{0,1}. \tag{4.4}$$

Notice that, as in the previous section, ω_6 is the only constant depending on the derivatives of the coefficients.

Proof. Let

$$\begin{aligned} A^* w &= \sum_{ij} D_j(a_{ij} D_i w) - \sum_j D_j(b_j w) - cw \\ &= \bar{A} w + \sum_{ij} (D_j a_{ij}) D_i w - \left(\sum_j D_j b_j \right) w. \end{aligned}$$

Then, from Theorem 3.6 it follows that A^* and $\bar{A}$ generate an analytic semigroup on $L^\infty(\mathbf{R}^n)$ and so the equation

$$(\lambda - A^*)w = \varphi \in L^\infty(\mathbf{R}^n), \qquad \operatorname{Re}\lambda \geqq \omega_5$$

has a unique solution $w \in D(A^*)$. Moreover, w satisfies

$$(\lambda - \bar{A})w = \varphi + \sum_{ij} (D_j a_{ij}) D_i w - \left(\sum_j D_j b_j \right) w$$

and so, by (3.10),

$$\begin{aligned} &\left\| \left[V \sum_j |D_j w|^2 \right]^{1/2} \right\|_{0,\infty} + \| Vw \|_{0,\infty} + |\lambda - \omega_5|^{1/2} \| w \|_{1,\infty,\sqrt{V}} \\ &\quad + |\lambda - \omega_5|\ \| w \|_{0,\infty} \leqq k_3 \left\| \varphi + \sum_{ij} (D_j a_{ij}) D_i w - \left(\sum_j D_j b_j \right) w \right\|_{0,\infty} \end{aligned} \tag{4.5}$$

where k_3, associated to $\bar{A}$, is independent of the derivatives of the coefficients. Furthermore, by (4.2) and (4.3),

$$\left\| \sum_{ij} (D_j a_{ij}) D_i w - \left(\sum_j D_j b_j \right) w \right\|_{0,\infty} \leqq k_4 (| w |_{1,\infty} + \varepsilon \| Vw \|_{0,\infty} + k_\varepsilon \| w \|_{0,\infty}) \tag{4.6}$$

for each $\varepsilon > 0$. From (4.5) and (4.6) we conclude that, if

$$\varepsilon = 1/(2k_3k_4), \quad \operatorname{Re} \lambda \geqq \omega_6 = \omega_5 + 2k_3k_4(2k_3k_4 + k_\varepsilon),$$

then

$$\begin{aligned} &\left\| \left[V \sum_j | D_j w |^2 \right]^{1/2} \right\|_{0,\infty} + \| Vw \|_{0,\infty} + | \lambda - \omega_5 |^{1/2} \| w \|_{1,\infty,\sqrt{V}} \\ &\quad + | \lambda - \omega_5 | \, \| w \|_{0,\infty} \leqq 2k_3 \| \varphi \|_{0,\infty}. \end{aligned} \tag{4.7}$$

Now, we can apply the method of Section 7.3 in [16] to obtain

$$\begin{aligned} \| u \|_{0,1} &= \sup \left\{ \int u(x)\varphi(x)dx : \varphi \text{ has compact support and } \| \varphi \|_{0,\infty} \leqq 1 \right\} \\ &\leqq \sup \left\{ \int u(\lambda - A^*)wdx : w \in D(A^*) \text{ satisfies (4.7), } \| \varphi \|_{0,\infty} \leqq 1 \right\} \\ &\leqq \sup \left\{ \int (\lambda - A)uwdx : w \in L^\infty(\mathbf{R}^n) \text{ satisfies (4.7), } \| \varphi \|_{0,\infty} \leqq 1 \right\}. \end{aligned} \tag{4.8}$$

in particular,

$$\| u \|_{0,1} \leqq 2k_3 | \lambda - \omega_6 |^{-1} \| (\lambda - A)u \|_{0,1}.$$

So, $(\lambda - A)$ is an injective operator with closed range in $L^1(\mathbf{R}^n)$ and (4.4) holds. The proof will thus be complete if we show that the range of $(\lambda - A)$ contains the space of functions $\varphi \in L^\infty(\mathbf{R}^n)$ with compact support, which is dense in $L^1(\mathbf{R}^n)$. Indeed, for such a function φ we have that

$$\Pi\varphi \in L^\infty(\mathbf{R}^n) \quad \text{where } \Pi(x) = \exp\sqrt{1 + | x |^2}.$$

Now, since $\pi(x) = \sqrt{1 + | x |^2}$ satisfies (3.15), by Remark 3.7 the equation $(\lambda - A)v = \varphi$ has a (unique) solution v such that

$$\Pi v \in C^1(\mathbf{R}^n, \sqrt{V}) \cap H^2(\mathbf{R}^n)_{\text{loc}}.$$

Therefore, $v \in \{ w \in L^1(\mathbf{R}^n) : Aw \in L^1(\mathbf{R}^n) \}$ and φ is in the range of $(\lambda - A)$. □

We now give two intermediate results.

4.2. LEMMA. *Assume* (4.1), (4.2), (4.3) *and* (2.3)–(2.6). *There exists* $k_7 > 0$, *which does not depend on the derivatives of the coefficients* a_{ij}, b_j *and* c, *and* $\omega_7 \geqq \omega_6$ *such that for* $\operatorname{Re}\lambda \geqq \omega_7$ *the solution* $u \in L^1(\mathbf{R}^n)$ *of the equation* $(\lambda - A)u = f$, *where* $f \in L^\infty(\mathbf{R}^n)$ *has compact support, satisfies*

$$\| Vu \|_{0,1} + |\lambda - \omega_6|^{1/2} \| \sqrt{V}u \|_{0,1} \leqq k_7(\| f \|_{0,1} + |u|_{1,1}) + 2^{-1} \left\| \left[V \sum_j |D_j u|^2 \right]^{1/2} \right\|_{0,1}. \tag{4.9}$$

PROOF. Arguing as we did to obtain (4.8), it follows that

$$\| Vu \|_{0,1} \leqq \sup \left\{ \int Vu(\lambda - A^*)w dx : w \in D(A^*) \text{ satisfies (4.7)}, \ \| \varphi \|_{0,\infty} \leqq 1 \right\}$$

$$\leqq \sup \left\{ \int (\lambda - A)(Vu) w dx : w \in L^\infty(\mathbf{R}^n) \text{ satisfies (4.7)}, \ \| \varphi \|_{0,\infty} \leqq 1 \right\}.$$

Now,

$$\int (\lambda - A)(Vu) w dx = \int Vw(\lambda - A)u dx - \int w \sum_{ij} a_{ij} D_j u D_i V dx + \int \sum_{ij} a_{ij} u D_j V D_i w dx - \int \sum_j b_j u D_j V w dx$$

and so, recalling (4.1) and (4.7), we have, by standard computations,

$$\| Vu \|_{0,1} \leqq k(\| f \|_{0,1} + |u|_{1,1}) + 2^{-1} \left\| \left[V \sum_j |D_j u|^2 \right]^{1/2} \right\|_{0,1},$$

the remainder of estimate (4.9) may be proved by a similar argument. □

4.3. LEMMA. *Assume* (4.1), (4.2), (4.3) *and* (2.3)–(2.6). *There exists* $k_8 > 0$, *which does not depend on the derivatives of the coefficients* a_{ij}, b_j *and* c, *and* $\omega_8 \geqq \omega_7$ *such that for* $\operatorname{Re}\lambda \geqq \omega_8$ *the solution* $u \in L^1(\mathbf{R}^n)$ *of the equation* $(\lambda - A)u = f$, *where* $f \in C^1(\mathbf{R}^n)$ *has compact support, satisfies*

$$\left\| \left[V \sum_j |D_j u|^2 \right]^{1/2} \right\|_{0,1} + |\lambda - \omega_6|^{1/2} |u|_{1,1} \leqq k_8 \| f \|_{0,1}. \tag{4.10}$$

PROOF. Arguing as in the previous proof we conclude that, for any $s = 1, \ldots, n$,

$$\begin{aligned}\| D_s u \|_{0,1} & \\ &\leqq \sup\left\{\int D_s u(\lambda - A^*) w dx : w \in D(A^*) \text{ satisfies } (4.7),\ \| \varphi \|_{0,\infty} \leqq 1\right\} \\ &\leqq \sup\left\{\int w(\lambda - A) D_s u dx : w \in C^1(\mathbf{R}^n)\ \textit{satisfies}\ (4.7),\ \| \varphi \|_{0,\infty} \leqq 1\right\}.\end{aligned}$$

Now,

$$\begin{aligned}\int w(\lambda - A) D_s u dx = &- \int D_s w(\lambda - A) u dx - \int \sum_{ij} D_s a_{ij} D_j u D_i w dx \\ &+ \int \sum_j D_s b_j D_j u w dx - \int D_s c u w dx.\end{aligned}$$

Thus, recalling (4.1), (4.2), (4.3) and (4.7) we obtain the "second half" of the desired inequality; the "first half" may be proved in the same way. □

4.4. Theorem. *Assume* (4.1) *and* (2.3)–(2.6). *Let* $a_{ij} \in C^1(\mathbf{R}^n)_u$. *There exists* $k_9 > 0$, *which does not depend on the derivatives of the coefficients* a_{ij}, *and* $\omega_9 \in \mathbf{R}$ *such that for* $\operatorname{Re} \lambda \geqq \omega_9$ *the equation*

$$(\lambda - A) u = f \in L^1(\mathbf{R}^n)$$

has a unique solution $u \in L^1(\mathbf{R}^n)$ *and*

$$\begin{aligned}&\left\| \left[V \sum_j | D_j u |^2 \right]^{1/2} \right\|_{0,1} + \| V u \|_{0,1} + | \lambda - \omega_9 |^{1/2} \| \sqrt{V} u \|_{0,1} \\ (4.11) \qquad &+ | \lambda - \omega_9 |^{1/2} | u |_{1,1} + | \lambda - \omega_9 | \, \| u \|_{0,1} \leqq k_9 \| f \|_{0,1}.\end{aligned}$$

Proof. From Lemmas 4.1, 4.2 and 4.3 we have that the conclusion of Theorem 4.4 holds under the additional conditions (4.2), (4.3) and assuming $f \in C^1(\mathbf{R}^n)$ with compact support. One can easily remove the last extra-hypothesis by an approximation procedure. As for (4.2), (4.3), we can use standard contraction techniques in the Banach space $\{u \in H^{1,1}(\mathbf{R}^n, \sqrt{V}) : \sqrt{V} u \in H^{1,1}(\mathbf{R}^n, \sqrt{V})\}$ (see the proof of Theorem 1.2 in [5]). □

The previous result may be extended to weighted L^1 spaces (see Remark 3.7).

5. Generation in $L^p(\mathbf{R}^n)$, $1 < p < \infty$

We consider an operator E of the form

$$Eu = \sum_{ij} a_{ij}D_{ij}u + \sum_j b_jD_ju - cu.$$

The following result may be easily deduced from Theorems 3.6 and 4.4.

5.1. Lemma. *Assume* (4.1) *and* (2.3)–(2.6). *Let* $a_{ij} \in C^1(\mathbf{R}^n)_u$. *There exists* $k_{10} > 0$, *which does not depend on the derivatives of the coefficients* a_{ij}, *and* $\omega_{10} \in \mathbf{R}$ *such that for* $\operatorname{Re}\lambda \geqq \omega_{10}$ *the equation*

$$(\lambda - E)u = f \in L^p(\mathbf{R}^n), \qquad 1 < p < +\infty$$

has a unique solution $u \in H^{1,p}(\mathbf{R}^n, \sqrt{V})$ *and*

$$\| u \|_{2,p,V} + |\lambda - \omega_{10}|^{1/2} \| u \|_{1,p,\sqrt{V}} + |\lambda - \omega_{10}| \, \| u \|_{0,p} \leqq k_{10} \| f \|_{0,p}. \tag{5.1}$$

Proof. From Theorems 3.6 and 4.4, via the Riesz–Thorin Interpolation Theorem, we obtain the existence and uniqueness of the solution $u \in H^{1,p}(\mathbf{R}^n, \sqrt{V})$ and the estimate

$$\begin{aligned} &\left\| \left[V \sum_j |D_ju|^2 \right]^{1/2} \right\|_{0,p} + \| Vu \|_{0,p} + |\lambda - \omega_{10}|^{1/2} \| \sqrt{V}u \|_{0,p} \\ &\quad + |\lambda - \omega_{10}|^{1/2} |u|_{1,p} + |\lambda - \omega_{10}| \, \| u \|_{0,p} \leqq k_{10} \| f \|_{0,p}. \end{aligned} \tag{5.2}$$

In order to recover the full estimate (5.1), we merely need to apply standard regularity results to the equation

$$\sum_{ij} a_{ij}D_{ij}u = f - \sum_j b_jD_ju + (c - \lambda)u \in L^p(\mathbf{R}^n). \qquad \square$$

The assumption $a_{ij} \in C^1(\mathbf{R}^n)_u$ in Lemma 5.1 may be removed by the contraction technique of [5], as we have already done in the previous section. We can therefore state the resulting generation Theorem.

5.2. Theorem. *Assume* (4.1) *and* (2.3)–(2.6). *There exists* $k_{11} > 0$, *which does not depend on the derivatives of the coefficients* a_{ij}, *and* $\omega_{11} \in \mathbf{R}$ *such that for* $\operatorname{Re}\lambda \geqq \omega_{11}$ *the equation*

$$(\lambda - E)u = f \in L^p(\mathbf{R}^n), \qquad 1 < p < +\infty$$

has a unique solution $u \in H^{1,p}(\mathbf{R}^n, \sqrt{V})$ *and*

$$\| u \|_{2,p,V} + |\lambda - \omega_{11}|^{1/2} \| u \|_{1,p,\sqrt{V}} + |\lambda - \omega_{11}| \, \| u \|_{0,p} \leqq k_{11} \| f \|_{0,p}.$$

The previous result may be extended to weighted L^p spaces (see Remark 3.7).

6. Application to parabolic equations

In this section we use our generation results to study the Cauchy problem

$$\begin{aligned} &u_t - E(t)u = f \in L^p(0, T; L^q(\mathbf{R}^n)), \\ &u(0) = 0, \end{aligned} \tag{6.1}$$

where $p, q > 1$ and

$$E(t) = \sum_{ij} a_{ij}(t, x) D_{ij} u + \sum_{j} b_j(t, x) D_j u - c(t, x)u.$$

We suppose, for simplicity, that $V(x) \geqq 1$ has linear growth as in (2.2) and we assume that operators $E(t)$ satisfy the following conditions uniformly for $t \in [0, T]$:

$$\sum_{ij} a_{ij}(t, x)\xi_i\xi_j \geqq \nu |\xi|^2, \tag{6.2}$$

$$\sum_{j} |b_j(t, x)|^2 \leqq \nu\alpha_0 B^2 V(x), \tag{6.3}$$

for some constants $\nu > 0$ and $B \in [0, 2[$ and

$$c(t, x) = \alpha(t, x)V(x) + \beta(t, x) \tag{6.4}$$

with $\alpha, \beta \in C([0, T] \times \mathbf{R}^n)_u$, $\alpha(t, x) \geqq \alpha_0 > 0$. Moreover, we assume that

$$a_{ij},\ b_j/\sqrt{V},\quad c/V \in C([0, T] \times \mathbf{R}^n)_u, \qquad \forall i, j = 1, \ldots, n. \tag{6.5}$$

If $f \in L^p([0, T] \times \mathbf{R}^n)$, then one can see [13] for a detailed analysis of problem (6.1) when $E(t)$ has bounded coefficients.

The case of unbounded coefficients has been treated by several authors, such as Eidel'mann [10]. The case of $f \in L^p(0, T; L^q(\Omega))$, where Ω is a bounded domain, has also been studied by von Wahl [18]. The approach we give below is, however, very different from the one of [18].

6.1. THEOREM. *Assume* (2.2) *and* (6.2)–(6.5). *Then there exists a unique solution* u *of problem* (6.1) *such that*

$$u_t \in L^p(0, T; L^q(\mathbf{R}^n)), \quad u \in L^p(0, T; H^{2,q}(\mathbf{R}^n, V)).$$

Moreover

$$\int_0^T (\| u_t \|_{0,q}^q + \| u \|_{2,q,V}^q)dt \leqq k \int_0^T \| f \|_{0,q}^q dt. \tag{6.6}$$

Proof. We split the proof of Theorem 6.1 in a few steps. To begin with, we note that for $p = q = 2$ the proof is standard via Fourier Transform with respect to t (see e.g. Theorem 3.2 of [14]). Moreover, one can extend this result to the weighted spaces considered in Remark 3.7: if

$$\pi(x) = \ln[\Pi(x)]$$

satisfies (3.15) and $\Pi f \in L^2(0, T; L^2(\mathbf{R}^n))$, then problem (6.1) has a unique solution u and

$$\int_0^T (\| \Pi u_t \|_0^2 + \| \Pi u \|_{2,V}^2)dt \leqq k \int_0^T \| \Pi f \|_0^2 dt. \tag{6.7}$$

Next, let $2 < p = q < 2^* = 2n/(n-2)$. Then, to localize our problem let $\{x_N\}_{N \geq 1}$ be a sequence of points with integer coordinates in $\mathbf{R}^n$ such that $\bigcup_N B(x_N, \sqrt{n})$ covers $\mathbf{R}^n$ and let $\theta_N \in C^\infty(\mathbf{R}^n)$ be a cut-off function such that $0 \leqq \theta \leqq 1$, $\theta \equiv 1$ on $B(x_N, \sqrt{n})$, $\theta \equiv 0$ out of $B(x_N, 2\sqrt{n})$ and $|D^\alpha \theta_N| \leqq k$ for $|\alpha| \leqq 2$ and $N \geqq 1$. Obviously, $u_N = \theta_N u$ is a solution of the problem

$$\begin{aligned} &(u_N)_t - E(t)u_N = g_N \in L^p(0, T; L^p(\mathbf{R}^n)), \\ &u_N(0) = 0, \end{aligned} \tag{6.8}$$

where

$$g_N = \theta_N f - \sum_{ij} a_{ij}(2D_i u D_j \theta_N + u D_{ij} \theta_N) - \sum_j b_j u D_j \theta_N. \tag{6.9}$$

Now, using the homothetical transformation $x \to x\sqrt{V(x_N)}$ and recalling (2.15), we can turn problem (6.8) into a problem with uniformly bounded coefficients and then obtain

$$\int_0^T (\| (u_N)_t \|_{0,p}^p + \| u_N \|_{2,p,V}^p)dt \leqq k \int_0^T \| g_N \|_{0,p}^p dt \tag{6.10}$$

where k is, of course, independent of N. Thus, by the Sobolev Theorem, from (6.9) and (6.10) we conclude that

$$\int_0^T (\| (u_N)_t \|_{0,p}^p + \| u_N \|_{2,p,V}^p)dt \leqq k \int_0^T (\| \theta_N f \|_{0,p}^p + \| u_N \|_{2,V}^p)dt. \tag{6.11}$$

Next, to estimate the right-hand side of (6.11), we consider a weight function $\Pi \in C^\infty(\mathbf{R}^n)$ such that $0 \leqq \Pi \leqq 1$ and

$$\Pi \equiv 1 \quad \text{on } B(0, 2\sqrt{n}), \qquad \Pi(x) = \exp(-\sqrt{1+|x|^2}) \quad \text{out of } B(0, 3\sqrt{n}).$$

Obviously, $\Pi_N(x) = \Pi(x - x_N)$ is an admissible weight for $E(t)$ and so from (6.7) we derive

$$\int_0^T \| \Pi_N u \|_{2,V}^2 dt \leqq k \int_0^T \| \Pi_N f \|_0^2 dt. \tag{6.12}$$

Moreover, by Holder's inequality.

$$\| \Pi_N f \|_0 \leqq \| \sqrt{\Pi_N} f \|_{0,p} \sqrt{\| \Pi_N \|_{0,q}} = k_p \| \sqrt{\Pi_N} f \|_{0,p} \tag{6.13}$$

where $q = p/(p-2)$. But

$$\| u_N \|_{2,V} \leqq k \| \Pi_N u \|_{2,V}. \tag{6.14}$$

So, (6.11)–(6.14) yield

$$\int_0^T (\| (u_N)_t \|_{0,p}^p + \| u_N \|_{2,p,V}^p) dt \leqq k \int_0^T (\| \theta_N f \|_{0,p}^p + \| \sqrt{\Pi_N} f \|_{0,p}^p) dt. \tag{6.15}$$

Adding together estimates (6.15), we obtain (6.6) for $2 < p = q \leqq 2^*$.

Now, if $2^* < p = q \leqq 2^{**}$, we repeat the argument above. Theorefore, iterating the previous procedure we prove (6.6) in the case of $p = q \geqq 2$.

Also, the case of $1 < p = q < 2$ may be treated by known duality techniques, that we will not repeat here.

Finally, the general case $1 < p,\ q < \infty$ follows from what we have just proved via the result of [4]. □

Using the same technique, one can also treat non-zero initial data from a suitable interpolation space.

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